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A general graphical procedure for finding motion centers of planar mechanisms

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Abstract

For an infinitesimal motion of a given planar mechanisms we can consider a relative motion center for each pair of rigid components. The location of these relative centers is completely determined for mechanisms with only one internal degree of freedom. We give a graphical procedure to find all relative centers for such a mechanism in a given (non-singular) position, and give an inductive proof that it is general (by means of Henneberg sequences). Besides the known intersection techniques with Aronhold–Kennedy lines and center lines of 2-dof subframeworks, we make use of a classical geometric construction due to J. Baracs. We also show how a graphical procedure for relative centers can be used to find new geometric descriptions for the special positions of some isostatic frameworks.

MSC: 51N05; 51A45

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1. Introduction

If an engineer designs a mechanism, or if an orthopedic physician examines a human body, he distinguishes several rigid components that can be moved relatively to each other. In order to understand the kinematic behavior of such a (biological or artificial) machine in a particular position it is important to know the *center* about which two components rotate relatively to each

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other in the observed or desired (infinitesimal) motion of the mechanism. For plane mechanisms, centers of relative motion are points; for spatial mechanisms, they are lines. When a relative motion is a translation, there is still a center, but it "lies at infinity." We refer to [14] for the fundamentals of mechanisms and (absolute or relative) centers of first order motion.

In this article we treat planar mechanisms, called (planar) *bar frameworks*, consisting of rigid bars that are connected to each other by universal joints. See [4] for a clear introduction to this subject. We focus on *1-underbraced* frameworks. Such frameworks have one internal degree of freedom, that is, they have four total degrees of freedom: two degrees of translation and one of rotation (these being rigid motions, moving the framework without flexing it), plus one additional degree of freedom, which we call "internal" due to the underbracing. See Figs. 4 and 5 for examples of (planar) 1-underbraced bar frameworks. These 1-underbraced frameworks appear in text books on mechanical engineering as *linkages*, where bars might be replaced by more general rigid components ("links"), and where often certain vertices are "pinned down" and constitute a "ground link" (e.g., [6,14,15]).

For these mechanisms, there is essentially only one way to flex them infinitesimally, up to a global multiple of the applied joint velocities, and up to Euclidean motions. Consequently, for a 1-underbraced framework, the geometric positions of the relative centers of motion for pairs of rigid components are determined. In mechanical engineering, these points are called *instantaneous centers of zero velocity* or briefly *instant centers*. We refer to [3,14] for a motivation to look for graphical procedures that construct these instant centers in a given position of the linkage.

The configuration of relative centers is constrained by the *Aronhold–Kennedy Theorem*: "If three rigid bodies in the plane are in motion relative to each other, then their three relative centers are collinear." In general, if n rigid bodies in the plane are in motion relative to each other, then their $\binom{n}{2}$ relative centers form a *generalized Desargues configuration* (Section 2).

If two bars or rigid components of a given framework are attached by a common joint, then they can only move relative to each other about a center at that joint ("primary instant centers"). In a 1-underbraced bar framework, rigid components that are not directly connected by a joint still have a relative center of motion at a well-determined geometric location ("secondary instant centers"). In Fig. 4 we see how the four joints of a closed chain of four bars determine the two remaining relative centers, each by applying the Aronhold–Kennedy Theorem twice. This simple trick is often used in graphical mechanics [4,14,15], and we refer to it as *the rule of four*. However, as shown in Fig. 5, the rule of four sometimes fails to construct relative centers of motion for a mechanism with only one internal degree of freedom. Indeed, the shown mechanism does not contain a cycle of 4 bars. In mechanical engineering, a linkage is called *indeterminate* if it prevents the construction of its instant centers by merely applying the "rule of 4" [3]. The "double butterfly linkage" is a famous example of an indeterminate linkage. In [3] the authors present a graphical technique that is able to locate the instant centers for a given (sufficiently general) position of this double butterfly linkage, but they do not mention whether and how their technique generalizes to arbitrary (indeterminate) linkages.

The main objective of this article is to give a general graphical procedure to construct all relative centers of motion in a given 1-underbraced bar framework in the plane. The constructions start with given positions for the joints of the 1-underbraced framework, as data, and proceed in stages, forming at each step the join (a line) of two known points or the meet (an intersection point) of two known lines. These two operations of elementary projective geometry do not quite suffice for a complete solution. We need one further operation from synthetic projective geometry, known as the "Baracs construction" (Section 6), due to Janos Baracs. This is enough: we arrive at a complete algorithm for finding the relative centers of motion of a 1-underbraced framework in the plane. The Baracs construction has proven to be an important tool in synthetic projective geometry; we are happy here to be able to introduce yet another of its applications. In order to guarantee that our geometric constructions never "degenerate" (joining coincident points or intersecting coincident lines) we will have to assume a general position for the given framework, in a sense that is given later in the article.

The algebraic analogue of this solution is a construction in the Grassmann algebra of order 3, over $\mathbb{R}$, generated by a set $a_1, \ldots, a_n$, variables in one–one correspondence with the joints of the framework, and a dummy variable z arising from the Baracs construction. In this purely algebraic context we arrive at a coordinate-free expression for the positions of the relative centers of motion.

The family of 1-underbraced plane frameworks can be built inductively from a simple "scissors" framework, using the Henneberg moves employed in [13]. Following the Henneberg construction, we obtain an inductive proof (Section 7) that the rule of four, a principle of controlled edge swapping, together with the Baracs construction, suffice to calculate the relative centers of motion of any 1-underbraced plane framework. The material of this article has been developed independent from (even unaware of) the method by [3]. To the best of our knowledge the use of the Baracs construction is an original contribution, yielding the first general graphical method for finding instant centers of mechanisms with one internal degree of freedom.

Finally, in Section 8 we offer an application of our algorithm. A graph is called *isostatic* if it can be realized as an infinitesimally rigid bar framework in the plane, and if moreover the deletion of any edge destroys this property. A *special position* of an isostatic graph is an embedding in the plane which makes the bar framework infinitesimally flexible. In [17] it is shown that, except for some singular configurations with coinciding joints, the special positions of an isostatic graph G are given as the zero set of one polynomial $C(G)$, called the *pure condition* of G. Also in [17] the authors show how to obtain the polynomial $C(G)$, as an element of the *Bracket Ring*, for an isostatic mechanism in any dimension n. Although the authors explain the geometric meaning of the pure condition $C(G)$ in many examples, a geometric understanding of the special positions in the general case seems to be missing. We suggest another approach. Given an isostatic graph G, delete an edge ab, forming a 1-underbraced graph $G - ab$. Our algorithm produces a relative center for two distinct components of $G - ab$, expressed synthetically in terms of the names of the vertices, the name of one variable point (for the Baracs construction), and the operators join and meet. This construction degenerates exactly when the original framework G is in special position, and fails to be isostatic. In this way, we arrive at a "Cayley factorization" of pure conditions for arbitrary graphs isostatic in the plane. Examples illustrate this procedure, yielding new geometric interpretations for special positions of some non-trivial isostatic graphs.

2. Centers of motion

For each (infinitesimal) Euclidean motion in the plane we can define a *center of motion* C, which is a vector in $\mathbb{R}^3$:

(1) For a translation with constant velocity $v = (v_1, v_2)$ we set

$$C = (-v_2, v_1, 0).$$

(2) For a rotation about $c = (c_1, c_2)$ with angular velocity α we set

$$C = (\alpha c_1, \alpha c_2, \alpha).$$

We also allow a translation with zero velocity, or a rotation with zero angular velocity, which means that nothing has been moved at all. For this (zero) motion we have a center C equal to $(0, 0, 0)$.

If $C \neq (0, 0, 0)$ then it can be regarded as a vector of homogeneous coordinates of points $\pi(C)$ in the projective extension of the work plane. Thus, for a rotation with center $C = (\alpha c_1, \alpha c_2, \alpha)$ the corresponding point $\pi(C)$ "sits at" the Euclidean center of rotation (c_1, c_2). For a translation with center $C = (-v_2, v_1, 0)$, $\pi(C)$ is the point "at infinity" (in "direction" $(-v_2, v_1)$, perpendicular to the constant velocity vector), and is said to "sit at" that infinite point. Note that the projective geometric point contains less information than the algebraic center C of motion, since it does not determine the magnitude of velocity, only its direction. (For kinematics in 3 dimensions, see [2,16], where centers of relative motion become vectors in $\mathbb{R}^6$.)

If two rigid bodies in the plane, B_1 and B_2, are subject to a (separate) infinitesimal motion, with centers C_1 and C_2 respectively, then we define the *relative center of motion* of B_2 w.r.t. B_1 as:

$$C_{12} = C_2 - C_1 \in \mathbb{R}^3.$$

If $C_1 \neq C_2$ then B_1 and B_2 move relative to each other, and $C_{12} \neq (0, 0, 0)$. In this case, C_{12} can be considered as homogeneous coordinates of a point in the projective plane. Since $C_{12} = -C_{21}$, these relative centers sit at the same point. If the bodies B_1 and B_2 are attached to each other by a common point (*hinge* or *joint*), and if they move relative to each other, then their relative center of motion always sits at this common joint.

An immediate but important consequence of the definition of relative centers is the following classical property (*Aronhold–Kennedy*):

If three rigid bodies in the plane are subject to an instantaneous motion, and if none of the three relative centers is zero, then the latter sit at three collinear points.

Indeed,

$$C_{12} + C_{23} + C_{31} = 0.$$

If we consider four bodies that move relatively to each other in the plane, then the six centers C_{ij} form a *complete quadrilateral*, shown on the left in Fig. 1. This is the figure formed by intersecting the six edges of a spatial tetrahedron with a plane. In fact, given four vectors $C_1, \ldots, C_4$ in 3-space, their six differences produce *coplanar* points that form such a complete quadrilateral. If the four original points C_i are taken to be the (projective) centers of motion of four bodies, then the resulting complete quadrilateral is the diagram of centers of relative motion.

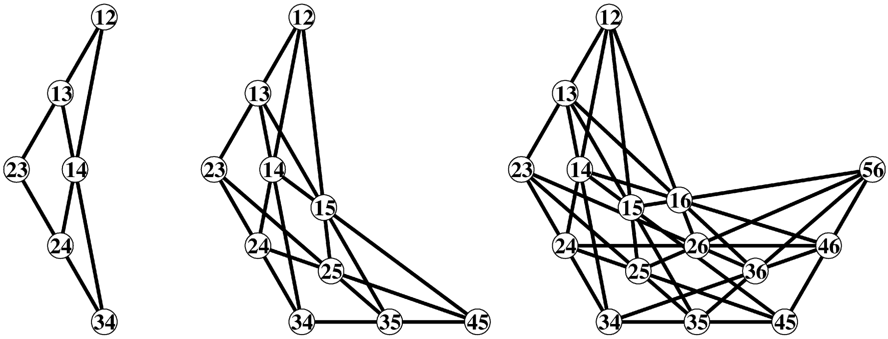

Fig. 1. Diagrams of relative centers of motion for 4, 5 and 6 bodies in the plane.

Given n vectors C_i, their differences C_{ij} form a generalized Desargues configuration ($n = 5$ yields the usual Desargues configuration of 10 points and 10 lines). These we show in Fig. 1 for $n = 4, 5, 6$.

Note also that if you drop the label '5' wherever it appears in the case $n = 5$, you find the combined diagram of absolute centers C_i and relative centers C_{ij} for a motion of four bodies. The four points with a singleton label form the vertices of a tetrahedron, while the other six points form a plane section with the edges of that tetrahedron.

The same figures appear elsewhere in mathematics as the diagram of reflections of a Coxeter group, for the symmetric group S_n, the Coxeter group denoted A_{n-1}. A combinatorially natural coordinatization in real projective space of rank n is by the integer vectors:

$$C_{ij} \to (0, 0, \dots, 0, 1, 1, \dots, 1, 0, \dots, 0)$$

where the 1's are in positions $i, i + 1, \dots, j - 1$. Thus, for D_4,

$$\begin{aligned} &C_{12} \to (1, 0, 0), \qquad C_{23} \to (0, 1, 0), \qquad C_{34} \to (0, 0, 1), \\ &C_{13} \to (1, 1, 0), \qquad C_{24} \to (0, 1, 1), \qquad C_{34} \to (1, 1, 1). \end{aligned}$$

When several bodies are joined together at a single point, the diagram of relative centers degenerates, but is still the projection of a generalized Desargues configuration. For instance, if, in a system of six bodies, bodies $1, 2, 3$ share a common vertex, the generalized Desargues configuration D_6 degenerates to the figure of fifteen points in Fig. 2. The centers C_{12}, C_{13}, C_{23} coincide at a point we have marked C_{123}, a point collinear with C_{14}, C_{24}, C_{34}.

The study of the (infinitesimal) motions of a planar mechanism with rigid components is dual to the problem of lifting a planar layout of lines to a spatial configurations. In this dual setting the center C_i assigned to a body B_i has the meaning of a forced spatial incidence between two lifted lines. So, some of the lifted lines will be coplanar in each legal lifting. These planes correspond to the rigid components of the primal mechanism. The relative centers correspond to lines of intersection of these planes. In [8,9] this dual problem has been treated, albeit with another purpose than this paper.

3. Bar frameworks

The concept of a bar framework can be explained at two levels, a combinatorial and a geometric one. At the combinatorial level, only the design of the framework is specified: the number

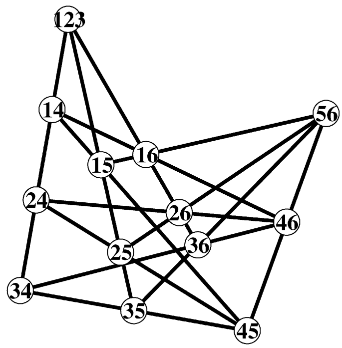

Fig. 2. A degenerated diagram of relative centers for 6 bodies in the plane. Indeed, $C_{12} = C_{13} = C_{23}$.

of joints and bars, and the pattern of incidences of bars with joints. From this point of view, a bar framework is given by a graph $G = (V, E)$ with vertex set V and edge set E (we exclude loops and multiple edges). At the geometric level, this design has been physically implemented: the bars now have a fixed (nonzero) length, and the joints are given a position in a Euclidean space $\mathbb{R}^n$ of some dimension n. This space will always be the plane $\mathbb{R}^2$ in this paper. So, a geometric framework is a pair $F = (G, P)$, where $G = (V, E)$ is a (simple) graph, and where

$$P : V \to \mathbb{R}^2$$

is a realization, with $P(a) \neq P(b)$ if $ab \in E$. Once implemented in $\mathbb{R}^2$, the edges are called the *bars*, and the vertices are called the *joints* of the framework. Sometimes P is left implicit in the notation and we write $a, b, \ldots$ for the joints as well. See Fig. 3 for a framework in the plane with 9 bars and 6 joints.

Motions or flexes of a bar framework must be described at the geometric level. We will not treat the difficult issue of finite motions here, but restrict to the infinitesimal (or instantaneous) motions. One way to represent such a motion is by assigning a velocity vector to every joint, respecting the condition that every bar is rigid. However, for our purposes it is more convenient to give the center of motion C_i for each bar e_i of F (Section 2). In this notation a motion γ of F is represented by a vector in $\mathbb{R}^{3e}$:

$$\gamma = (C_1, \ldots, C_e)$$

where e is the number of bars of F. Notice that a labeling of the bars (or edges of G) is needed.

Analogous to the bar conditions for joint velocities, the centers of motion of bars in a framework are subject to *joint conditions*. Indeed, if a two edges e_i and e_j are attached by a common

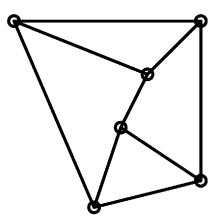

Fig. 3. A bar framework in the plane. The structure graph has 9 edges and 6 vertices ($e = 2v - 3$).

joint $p_{ij} = (x_{ij}, y_{ij})$, then p_{ij} should undergo the same motion both as point of e_i and as point of e_j:

$$C_i \vee (x_{ij}, y_{ij}, 1) = C_j \vee (x_{ij}, y_{ij}, 1) \quad \Longleftrightarrow \quad (C_i - C_j) \vee (x_{ij}, y_{ij}, 1) = \mathbf{0}.$$

Here, the symbol $\vee$ denotes the join operator in the Grassmann algebra (or the exterior product, or the cross product). Consequently, one joint condition comes down to two linearly independent equations in six unknown coordinates of C_i and C_j.

If the (infinitesimal) motion uses the same center C for each bar,

$$C_1 = \cdots = C_e = C$$

then the motion is called *trivial* or *Euclidean*. If a framework $F = (G, P)$ only allows Euclidean motions then it is *infinitesimally rigid*. A *body* of a bar framework F is a maximal infinitesimally rigid subframework of F. The set of bodies defines a partition of the edge set of a framework. To specify a motion γ it suffices to give a center of motion for each body of the framework.

Let G be a given graph (design). If G has a realization as an infinitesimally rigid framework F in $\mathbb{R}^2$, then G itself is called infinitesimally rigid.[1] We can also consider the notion of "bodies" directly for graphs. If a graph G is infinitesimally rigid, but if the removal of any edge causes G to lose this property, then we call such a graph *isostatic*. For example, the underlying graph of the bar framework in Fig. 3 is isostatic. Isostatic graphs (in dimension 2) have been combinatorially characterized by Laman in 1970 [5].[2] As a matter of fact, since 1970, other equivalent characterizations have been deduced by several authors. See [4,13] for an overview.

Laman's count. A graph G represents a 2-isostatic bar framework if and only if $e = 2v - 3$ and for each subgraph G', $e' \leqslant 2v' - 3$. Here, e (respectively e') stands for the number of edges in G (respectively G'), and v (respectively v') for the number of vertices in G (respectively G') [5].

The isostatic graphs on v vertices form the *bases* of a matroid on the set of all $\binom{v}{2}$ possible edges, the *Laman matroid*. If $G = (V, E)$ is a graph on v vertices and if ab is a missing edge of G that depends on E in the Laman matroid, then ab is called an *implicit edge*. If G is infinitesimally rigid then each possible edge ab either is present in G, either is implicit. It is a well-known property of the Laman matroid that the implicit edges coincide with the missing edges in the separate bodies. We refer to [7] for fundamentals of matroid theory.

A realization P of a graph G is called *generic* if the bodies of G according to the Laman matroid match the bodies of the framework $F = (G, P)$. In a generic realization $F = (G, P)$ of G the joints $P(a)$ and $P(b)$ corresponding to an implicit edge ab maintain their relative distance during any motion.

4. Underbraced frameworks

If for every plane realization P of a graph G the framework $F = (G, P)$ has a motion space of dimension at least 4 then this framework design (G) is called *underbraced*. More precisely, let $3 + k$ be the minimal dimension of the motion space for all realizations of G, then we say that the

1 It is well known that if G is infinitesimally rigid, then almost every realization F of G is infinitesimally rigid.
2 There is still no such proven characterization in 3-space.

framework is k-underbraced. A k-underbraced framework has a motion space of dimension $k+3$ in every generic realization: 3 Euclidean degrees of motion and k internal degrees of freedom.

Let $F=(G,P)$ be an underbraced bar framework, generically implemented in the plane, and let B_1 and B_2 be bodies of F. Then there exists an infinitesimal motion of F such that B_1 and B_2 move relatively to each other. During such a flex of F, the bodies B_1 and B_2 do not undergo the same Euclidean motion. For a 1-underbraced framework we can be more specific: if we flex such a framework infinitesimally, then every pair of bodies is in relative motion to each other.

Theorem 1. *If $F=(G,P)$ is a generic realization in the plane of a* 1*-underbraced graph G, and if we apply a non-trivial motion to F, then every two bodies in F move relative to each other.*

Proof. Let $\gamma=(C_1,\ldots,C_b)$ be any non-trivial motion for F (b: number of bodies). Because the motion space of F has dimension 4, it is generated by γ and the trivial motions $(C,C,\ldots,C)$. Suppose $C_i=C_j$ in γ, then in every motion of F the centers of B_i and B_j are equal. But this implies that B_i and B_j are contained in a larger rigid component of F, which contradicts the definition of body. □

If two bodies B_i and B_j of an underbraced framework F are attached to each other by a joint, then for every motion γ of F the relative center of motion $C_{ij}=C_j-C_i$ is geometrically determined (joint condition!), unless $C_i=C_j$. For 1-underbraced frameworks this generally holds for any pair of bodies, regardless a common joint or not. This property has been stated and proven in [8], albeit in the different (but equivalent) context of "almost-planar line configurations in 3-space."

Theorem 2. *If $F=(G,P)$ is a generic realization in the plane of a* 1*-underbraced graph G, then for each pair of bodies B_i and B_j there exists a point p_{ij} in the* (*projective*) *plane such that, for each non-trivial motion $\gamma=(C_1,\ldots,C_b)$ of F, the relative center C_{ij} sits at p_{ij}.*

Proof. Recall from the previous proof that the linear motion space of F is generated by the trivial motions and 1 non-trivial motion γ. If γ_1 is another arbitrary non-trivial motion of F, then $\gamma_1=\gamma_0+s\gamma$, with γ_0 some trivial motion, and s a nonzero real number. Under this motion, we get the following relative center for B_i and B_j:

$$(C+sC_j)-(C+sC_i)=s(C_j-C_i)=sC_{ij}$$

which sits at the same projective point as C_{ij}. □

Example. The *quadrangle Q* is a framework with 4 bars which are consecutively joined together to form a 4-gon, avoiding collinear bars. So, the structure graph equals C_4. One can see that Q is 1-underbraced (Fig. 4).

Let B_1,B_2,B_3,B_4 be the order in which the bars appear during a counterclockwise tour. Then, the relative centers $C_{B_1B_2},C_{B_2B_3},C_{B_3B_4},C_{B_4B_1}$ coincide with the joints of Q. In order to construct the remaining centers $C_{B_1B_3}$ and $C_{B_2B_4}$, we use the fact that the three relative centers of motion of three bodies in the plane are always collinear (Aronhold–Kennedy Theorem):

$$C_{B_1B_3}=(C_{B_1B_2}\vee C_{B_2B_3})\wedge(C_{B_1B_4}\vee C_{B_4B_3}),$$
$$C_{B_2B_4}=(C_{B_2B_1}\vee C_{B_1B_4})\wedge(C_{B_2B_3}\vee C_{B_3B_4}).$$

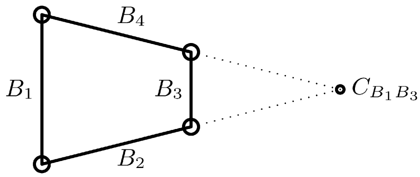

Fig. 4. The quadrangle Q. Once the framework has been constructed and positioned, the relative center $C_{B_1 B_3}$ (or $C_{B_2 B_4}$) is fixed, although it is not present as a joint!

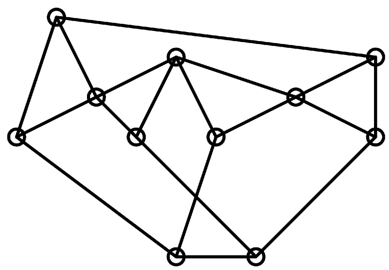

Fig. 5. In this 1-underbraced framework the rule of four cannot be applied to construct the relative center of motion for any pair of unhinged bodies.

(Here and elsewhere in this article, the Grassmann operators $\wedge$ ("meet") and $\vee$ ("join") can be interpreted as graphical operators, replacing a center C by its geometric point $\pi(C)$.)

Also in more complicated frameworks we will have the occasion to apply this principle, as soon as we can detect a cycle of 4 bodies with known relative centers for each consecutive pair of bodies. We will refer to this principle as *the rule of four*.

Unfortunately, not in every 1-underbraced framework the centers of relative motion can be constructed by means of the rule of four only.[3] The framework of Fig. 5 with 11 joints, 18 bars and 10 bodies lacks the presence of a cycle of bodies of length four. In this context, mechanical engineers sometimes use the term *indeterminate linkage*. The smallest example is the "double butterfly linkage" with eight links [3].

The main objective of this article is to give a general algorithmic construction in the projective plane for the relative centers of motion in a 1-underbraced bar framework. We stress the fact that the constructions given in [8] can be applied for 1-underbraced frameworks as well (after dualization), but that they are only valid for a restricted class of frameworks (so-called *simple frameworks*). Also, the techniques described in [3] seem promising, but till now they have only been used to particular cases and so far it is not clear whether they are generally applicable.

For certain arguments in the next sections, it is useful to have a sort of converse of Theorem 2.

Theorem 3. *Let A and B be bodies of a graph G and let F be a framework that generically realizes G. If there exists a point p_{AB} in the* (*projective*) *plane such that each motion γ of F that moves A relative to B has its relative center sitting at p_{AB}, then A and B are contained in a* 1*-underbraced subframework of F* (*or G*).

[3] Although this seems to be tacitly assumed in [6].

Proof. Because A and B are not contained in a larger body, there exist a vertex a of A and a vertex b of B such that ab is not an edge of G. Furthermore, ab is even not implicit in G.

Let γ be a motion of $F + ab$ (the extension of the framework F by bar ab). So,

$$(\gamma_A - \gamma_B) \vee a \vee b = 0.$$

Assume that γ moves A relative to B. Then $\gamma_A - \gamma_B$ sits at p_{AB}, and consequently:

$$(\mu_A - \mu_B) \vee a \vee b = 0$$

for each motion μ of F, which contradicts the fact that ab is not implicit in F. We conclude that $\gamma_A = \gamma_B$ for each motion γ of $F + ab$, and hence that A and B belong to a common body of $F + ab$, which proves the statement. □

For a given 2-underbraced framework in the plane, the relative center of motion of two bodies might not be determined anymore. Only when two bodies happen to belong to a 1-underbraced substructure of the given framework, the relative center of motion is fixed for each generic realization. If not, two bodies in a 2-underbraced framework always determine a line which necessarily contains their relative center when we flex the framework, the *relative center line*.

Theorem 4. *Let B_i and B_j be bodies of a 2-underbraced graph G that do not a belong to a common 1-underbraced subgraph, and let the framework F generically realizes G. Then there exists a line l_{ij} such that for each infinitesimal motion γ of F with nonzero relative center C_{ij} for B_i and B_j, C_{ij} sits at a point on l_{ij}.*

Proof. Theorem 3 guarantees the existence of non-trivial motions γ_1 and γ_2 of F such that they both cause a relative motion of B_i and B_j, but with relative centers C^1_{ij} and C^2_{ij} that sit at different points: $p^1_{ij} \neq p^2_{ij}$. Observe that this implies that γ_1 and γ_2 do not differ by a Euclidean motion. Now we put

$$l_{ij} = p^1_{ij} \vee p^2_{ij}.$$

Since F is the generic realization of a 2-underbraced graph, each motion γ of F can be written as

$$\gamma = \gamma_0 + r\gamma_1 + s\gamma_2$$

with γ_0 a Euclidean motion. But this means that the relative center of B_i and B_j in this motion γ can be expressed as

$$C_{ij} = rC^1_{ij} + sC^2_{ij}$$

which clearly sits at a point on l_{ij}. □

Remark. From an algorithmic point of view the relative center line for two bodies B_i and B_j of a generic 2-underbraced framework F can be constructed if one knows how to obtain the positions of the relative centers in certain (1-underbraced) bracings of F. Choose vertices a and

b such that ab is not an edge of F, even not implicitly. Further, choose c and d such that cd is not implicit in the extended framework $F + ab$. Then each non-trivial motion of $F_1 = F + ab$ can serve for γ_1 and each non-trivial motion of $F_2 = F + cd$ can serve for γ_2 as they appear in the proof of Theorem 4. More precisely:

Property 5. *If C^1_{ij} and C^2_{ij} are the relative centers of B_i and B_j in the bracings $F_1 = F + ab$ and $F_2 = F + cd$ respectively, then*

- *either they sit at the same point: $p^1_{ij} = p^2_{ij}$, which implies that B_i and B_j belong to a common 1-underbraced subframework of F;*
- *or the relative center line l_{ij} is determined by p^1_{ij} and p^2_{ij}.*

Proof. Because cd is not implicit in F_1, and as γ_1 is a non-trivial motion of F_1, there are bars cx and dy in F_1 that are in a relative motion under γ_1, and such that

$$\big(\gamma_1(cx) - \gamma_1(dy)\big) \vee c \vee d \neq 0.$$

On the other hand,

$$\big(\gamma_2(cx) - \gamma_2(dy)\big) \vee c \vee d = 0.$$

This implies that γ_2 cannot be written as $r\gamma_1 + \gamma_0$, with γ_0 a trivial motion. So, the motion space of F (modulo the trivial motions) is generated by γ_1 and γ_2. Consequently, if $C^1_{ij} \sim C^2_{ij}$ then for each other flexing $\gamma_3 = r\gamma_1 + s\gamma_2$ of F: $C^1_{ij} \sim C^3_{ij}$. This implies that B_i and B_j belong to a common 1-underbraced framework (Theorem 3). □

In order to construct certain relative centers C_{ij} of a given generic 1-underbraced framework F, we will apply the previous technique on the 2-underbraced framework $F - xy$ (deleting the edge xy from F), under the assumption that B_i and B_j are maintained in $F - xy$. Indeed, if well-chosen bracings $F_1 = (F - xy) + ab$ and $F_2 = (F - xy) + cd$ allow the construction of certain relative centers more easily, then we can construct the relative center line l_{ij} in $F - xy$. From this we have obtained a linear constraint for the position of C_{ij} in F, namely $\pi(C_{ij}) \in l_{ij}$. This is called the *swap principle*. In [3,10,11] the existence of the relative center line has been observed as well, albeit in the ad-hoc analysis of particular linkages, where it seems to play a crucial role in graphical methods.

5. Inductive constructions

There are some known procedures to build a larger 1-underbraced graph from a given one. This is accomplished in essentially the same way as extending isostatic graphs [13].

(1) *Henneberg move of type 1*: If G is a 1-underbraced graph, with a and b two vertices of G, then the introduction of a new vertex p which is linked to a and b, yields a 1-underbraced graph G^+ with one more vertex than G (also called a *2-valent* or *simple extension*).
(2) *Henneberg move of type 2*: If G is a 1-underbraced graph with edge ab, then the introduction of a new vertex p which is linked to a, b and some third vertex of G, combined with the

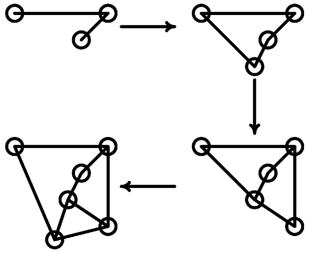

Fig. 6. A Henneberg construction starting from a scissor. The result is the isostatic graph of Fig. 3 with 1 bar deleted, hence 1-underbraced. The first two extensions are of type 1, the last is of type 2.

removal of the edge ab, yields a 1-underbraced graph G^+ with one more vertex than G (also called a *3-valent extension* or *edge splitting*).

The Henneberg moves really result in new 1-underbraced graphs, as can be shown by a combinatorial proof, using the same arguments as in [13] (Laman's count).

Our aim is to deduce "inductive constructions" for the relative centers of motion of 1-underbraced frameworks. This means that we use the relative centers of a given 1-underbraced framework G to construct those of the extended framework G^+. Natural choices for initial configurations in such construction sequences are a "scissor" (two bars connected by one joint) or a quadrangle. In [8] such inductive constructions are explained in the case of 2-valent extensions, and in the case of "compound extensions." In case of Henneberg moves of type 2 however, things are more complicated, because the removal of an edge in G causes some of the known relative centers to change. See Fig. 6.

Unfortunately, not every 1-underbraced graph can be built from a scissor by 2-valent and compound extensions only. But on the other hand, both Henneberg moves appear to form a generating set. The following theorem is a variant of a famous property of 2-isostatic graphs (see [13]).

Theorem 6. *Each* 1*-underbraced graph with independent edge set can be obtained from a scissor by means of a sequence of Henneberg constructions.*

Proof. We can add an edge to a 1-underbraced graph such that it becomes isostatic. We know that there exists a Henneberg sequence for this isostatic graph. Furthermore, Proposition 3.6 in [13] implies that we can start this Henneberg sequence with any edge from the graph. If we choose to start with the added edge, we can easily turn the Henneberg sequence into a construction for the 1-underbraced graph by merely deleting the first step. □

A *simple* 1-underbraced graph is a graph that can be obtained from a scissor by means of Henneberg extensions of the first type only (adding 2-valent vertices). In [8] it is described how to construct the relative centers of motion for a generic realization of a simple 1-underbraced graph. However, in [8] the constructions have been developed in the dual setting of "almost flat line configuration." Furthermore, degenerate positions for the rule of four have been neglected in [8]. Indeed, if the four relative centers that participate in the rule of four are all collinear then the rule of four cannot be applied (the meet of two coincident lines). Below, we give the complete argument, directly stated in the setting of simple 1-underbraced frameworks.

Theorem 7. *Let F be a 1-underbraced framework such that the position of the relative center C_{ij} is known for each pair of bodies B_i and B_j. Let F^+ be a framework obtained from F by adding a 2-valent joint p, connected with joints a and b of F, such that p, a and b are not collinear. Then, the "new" relative centers for F^+ are just joints or can be constructed from the known relative centers by applying the rule of four. In particular, the relative centers for a generic simple 1-underbraced framework can be constructed by merely applying the rule of four.*

Proof.

Case 1. Joints a and b share the same body B in F. Then, $B^+ = B + pa + pb$ is a body in G^+, and the relative centers of F^+ are essentially the same as those of F, putting

$$C_{AB^+} = C_{AB}$$

for any other body A in F^+ (or F).

Case 2. Joints a and b belong to bodies A and B respectively, and $A \neq B$. Then, bars pa and pb are bodies in F^+, and every body in F remains a body in F^+. Furthermore, if a pair of bodies in F^+, B_1 and B_2, was already present in F, the known construction for $C_{B_1 B_2}$ remains valid. Also, the relative centers $C_{pa,pb} = p$, $C_{pa,A} = a$ and $C_{pb,B} = b$ are directly available (as joints). Further, we construct:

$$C_{pa,B} = (p \vee b) \wedge (a \vee C_{AB}),$$
$$C_{pb,A} = (p \vee a) \wedge (b \vee C_{AB}).$$

These constructions never degenerate because $p \notin ab$.

Next, let D be a body in F different from A and B. Because F is a generically 1-underbraced, not all joints lie on the same line. In particular, there are two bodies D and E of F such that C_{DE} does not sit at the line ab. This implies that for at least one X in $\{D, E\}$ it holds that C_{AX} and C_{BX} do not both sit at ab. Say, $X = D$. Then at least one of the following constructions does not degenerate:

$$C_{pa,D} = (a \vee C_{AD}) \wedge (C_{pa,B} \vee C_{BD}),$$
$$C_{pb,D} = (b \vee C_{BD}) \wedge (C_{pb,A} \vee C_{AD}).$$

If the given construction for $C_{pb,D}$ happens to fail, the following alternative certainly works:

$$C_{pb,D} = (p \vee C_{pa,D}) \wedge (C_{AD} \vee C_{pb,A}).$$

If there happens to be a body E of F such that both C_{AE} and C_{BE} sit at line ab, then we can apply the rule of four by means of the previously constructed $C_{pa,D}$ and $C_{pb,D}$ to obtain $C_{pa,E}$ and $C_{pb,E}$:

$$C_{pa,E} = (a \vee C_{AE}) \wedge (C_{pa,D} \vee C_{DE}),$$
$$C_{pb,D} = (b \vee C_{BE}) \wedge (C_{pb,D} \vee C_{DE}).$$

Finally, the proof is finished by the property that the generic realizations of simple framework are given by performing the corresponding simple Henneberg sequence while always avoiding the collinearity of the added joint p with the attaching joints a and b. □

The main goal of this article is to find an inductive construction for the relative centers of motion in F^+ if this framework has been obtained from F by the second Henneberg construction. To this end, we need a non-trivial geometric construction, that is explained in the next section.

6. The Baracs construction

Suppose we are given a drawing of three collinear points in the projective plane: a, b and c. Furthermore, three arbitrary lines are given in that same plane: P, Q and R. Now we are asked to construct three lines A, B and C with the following properties:

$$a \in A, \qquad b \in B, \qquad c \in C,$$
$$A \wedge B \in R, \qquad B \wedge C \in P, \qquad A \wedge C \in Q.$$

Of course, if $\{P, Q, R\}$ happen to be concurrent lines, meeting in a point s say, then we can put $A = as$, $B = bs$, $C = cs$, yielding an easy solution. So, let us assume that P, Q and R are not concurrent (Fig. 7).

The solution we present here is based on the dual problem:

For three given concurrent lines a, b, c, and three given (non-collinear) points P, Q, R, find by construction three points A, B, C such that

$$A \in a, \qquad B \in b, \qquad C \in c,$$
$$R \in A \vee B, \qquad P \in B \vee C, \qquad Q \in A \vee C.$$

These two problems are polar to each other, and both they do not seem to allow a constructive solution at first sight. Where should we start with a geometric construction? Fortunately, this

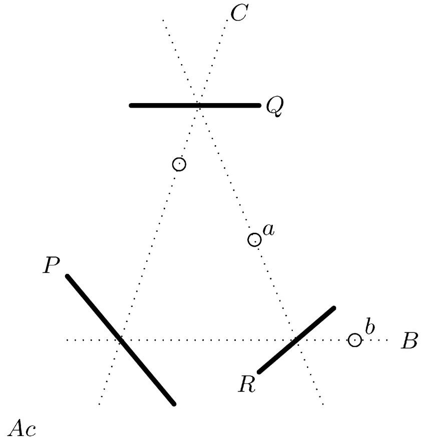

Fig. 7. Three collinear points a, b, c and three non-concurrent lines P, Q, R are given. We are asked to construct lines A, B and C through a, b and c respectively, such that their pairwise intersections lie on the given lines.

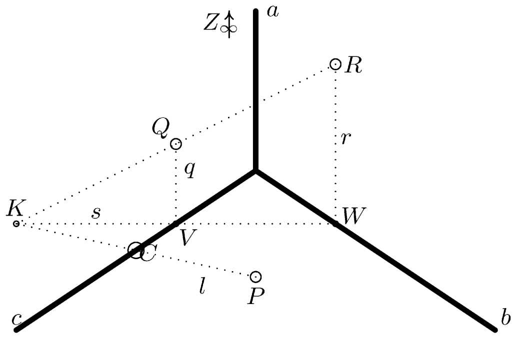

Fig. 8. The auxiliary point Z on the line a has been chosen at infinity. The plane QRZ intersects the lines c and b in V and W, respectively. The point K, common to QR and VW, must be the point of intersection of line QR with the bc-plane. And hence, the intersection of KP with line c must be the point C where plane PQR intersects c.

problem is a classical one, with a known solution (*Cours de Topologie Structurale*, Université de Montréal, 1978, by Janos Baracs, see also [1]). Indeed, we can regard the given drawing as a projection of a 3D scene, where $\{a, b, c\}$ represents a reference frame, and P, Q, R points in the planes bc, ac, ab, respectively. In this setting, we are looking for the points of intersection of the plane determined by P, Q, R with the three frame axes. See Fig. 8 for an instance of this problem, where the construction of point C is shown (from which A and B follow immediately).

One way to obtain these points of intersection goes by means of an auxiliary point Z on line a, which enables us to consider the auxiliary plane α through Z, Q, R (in Fig. 8 the point Z happens to lie at infinity). The line s of intersection of the planes α and bc is a straightforward construction:

(1) $Z \vee Q$ intersects c in V,
(2) $Z \vee R$ intersects b in W,
(3) $s = V \vee W$.

From this we get the point K of intersection of line $Q \vee R$ with plane bc:

(4) $K = s \wedge (Q \vee R)$.

Now we have the line l of intersection of plane PQR with plane bc:

(5) $l = K \vee P$.

Finally, we finish the job:

(6) $B = l \wedge b$,
(7) $C = l \wedge c$,
(8) $A = (C \vee Q) \wedge a$ of $(B \vee R) \wedge a$.

In order to have a construction which solves the primal problem, we just dualize the construction which solves the dual problem. Consequently, we add an auxiliary line Z through point a to the given drawing. So, we get:

(1) $V = c \vee (Z \wedge Q)$,
(2) $W = b \vee (Z \wedge R)$,
(3) $s = V \wedge W$,
(4) $K = s \vee (Q \wedge R)$,
(5) $l = K \wedge P$,
(6) $B = l \vee b$,
(7) $C = l \vee c$,
(8) $A = (C \wedge Q) \vee a$ of $(B \wedge R) \vee a$.

Of course, as well in the given drawing, as in the construction process, we allow points or lines to lie at infinity. In the remainder of this paper we refer to this solution as the *Baracs construction*.

We note that this construction is not *intrinsic*, meaning that it does not proceed simply by the successive formation of meets of pairs of lines, and joins of pairs of points, starting from the initial data of point and line positions. It was essential at one stage to introduce an arbitrary line passing through a given point. Since we will call this procedure repeatedly in the inductive process of determining relative centers of motion in a 1-underbraced framework, the final construction may eventually contain many such "general" (arbitrary) lines. However, it suffices to introduce one arbitrary auxiliary point z, such that every time we need a general line Z through a (given or constructed) point p, we can choose $Z = p \vee z$.

7. Adding a 3-valent vertex

Let G_n be a 1-underbraced graph with n vertices. We can assume that G_n does not contain dependent edges (otherwise we delete them). Choose three different vertices of G_n, v_1, v_2 and v_3, with at least one edge between them, say $Z = v_1v_2$. If we add a new vertex to G_n, v_{n+1}, and connect it with v_1, v_2 and v_3, and if moreover we delete edge Z, then we obtain a new 1-underbraced graph G_{n+1}, now with $n + 1$ vertices (Henneberg extension of type 2). We will use the following notations for the new edges: $D = v_1v_{n+1}$, $E = v_2v_{n+1}$ and $F = v_3v_{n+1}$ (Fig. 9).

In this section we will explain in full detail how to construct the relative center of motion for any pair of bodies in G_{n+1}, assuming known constructions for the relative centers of G_n or any other graph with at most n vertices. By Property 5 this also implies the knowledge of the relative center lines for graphs with at most n vertices.

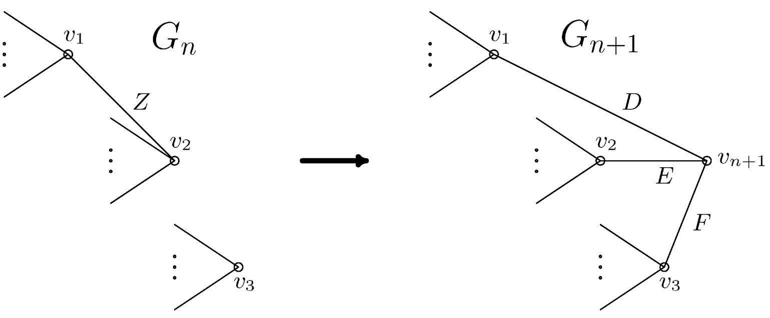

Fig. 9. Deleting edge Z from G_n, and adding a 3-valent vertex v_{n+1} to v_1, v_2, v_3, turns the 1-underbraced graph G_n into a new 1-underbraced graph G_{n+1}.

As in Section 5 the constructions are stated as symbolic join/meet-formulas using variables for the participating joints. This can be formalized in the Grassmann algebra. As soon as G_{n+1} is generically realized in the plane, the formulas become real geometric constructions.

We will always verify that the constructions do not degenerate (join of coinciding points or meet of coinciding lines). In order to guarantee this we assume the existence of a *non-degenerated Henneberg sequence* for the given 1-underbraced framework: in each Henneberg move of the sequence the new (2-valent or 3-valent) joint p is not collinear with any two of its neighbor joints. While this condition automatically holds in a generic realization as long as simple extensions are involved, it might not be fulfilled for Henneberg moves of type 2. We call such generic 1-underbraced frameworks *non-degenerated*.

Case 1. Assume that v_1, v_2 and v_3 belong to the same body B of G_n.

The construction of G_{n+1} comes down to a Henneberg move of type 2 on the isostatic subgraph B, resulting in a larger body $\overline{B}$ of G_{n+1}. So, the enlarged framework has essentially the same motion space as the original one, with the same relative centers if a non-trivial motion is applied.

Case 2. Assume that neither $\{v_1, v_3\}$ nor $\{v_2, v_3\}$ share the same body in G_n.

This implies that the bars D, E and F are separate bodies in G_{n+1}. We will treat the bodies of G_{n+1} in three different classes:

first class: the bodies (bars) $\{D, E, F\}$;
second class: the bodies incident with v_1, v_2 or v_3 but not one from $\{D, E, F\}$;
third class: the remaining bodies.

- First of all, notice that v_{n+1} is the relative center of motion for any pair of bodies of the first class.
- Next, consider a body A of the second class together with a body of the first class, say E (Fig. 10). Let us construct the relative center C_{AE}. If A and E have v_2 in common, then C_{AE} clearly sits on v_2. So we may assume that A does not contain v_2. Let us agree that A contains v_1.

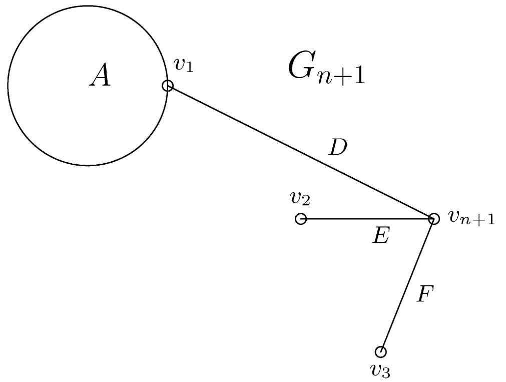

Fig. 10. In Case 2: if A is a body of G_{n+1} that contains v_1, but that is different from D, then A does not contain v_2 and v_3.

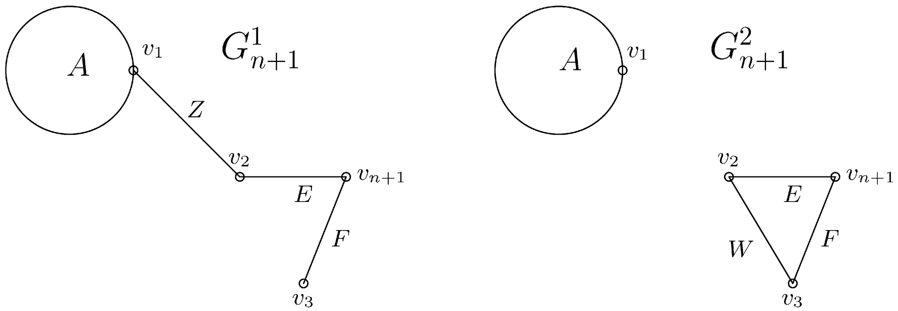

Fig. 11. The 1-underbraced graphs G^1_{n+1} and G^2_{n+1} are two separate extensions of the 2-underbraced graph $G_{n+1} - D$ (Fig. 10).

The key idea is to consider the 2-underbraced graph $G^-_{n+1} = G_{n+1} - D$. Notice that A and E are bodies in G^-_{n+1}. Further we can assume that A and E do not belong to a common 1-underbraced subgraph of G^-_{n+1}, otherwise the construction of C_{AE} could have been accomplished by induction. So, A and E define a relative center line in G^-_{n+1}, l_{AE}. We construct l_{AE} by means of the swap principle. This means that we look at two bracings of G^-_{n+1}, different from G_{n+1}: $G^1_{n+1} = G^-_{n+1} + Z = G_n + E + F$ and $G^2_{n+1} = (G_{n+1} - D) + W$, where W is an edge connecting v_2 and v_3 (Fig. 11). By the assumption for Case 2, the edge W was not present in G_n, even not implicitly. This means that Property 5 applies and that l_{AE} can be obtained by joining the relative centers C^1_{AE} and C^2_{AE} that are constructed in the separate bracings. Furthermore, these centers can be constructed by Theorem 7 in Section 5, because G^1_{n+1} is a 2-valent extension of G_n and G^2_{n+1} is a 2-valent extension of $G_n - Z + W$ (Fig. 11), both non-degenerated.
Finally, in G_{n+1} the relative center C_{AE} must lie on the line v_1v_{n+1}. It can be seen that in a non-degenerated realization, where v_1, v_2 and v_{n+1} are not collinear, $C^1_{AE} \notin v_1v_{n+1}$ (because $C^1_{AE} \neq v_1$, and $C^1_{AE} \in v_1v_2$), so $l_{AE} \neq v_1v_{n+1}$. We conclude that

$$C_{AE} = l_{AE} \wedge v_1v_{n+1}.$$

Similarly, we can construct C_{AD} and C_{AF} (for each body A of class 2).

– Next, we consider two bodies of the second class, A and B. Assume that A contains v_1 and that B contains v_2. According to the assumption for Case 2, A and B are not the same body. We intend to construct C_{AB}.
In the previous paragraph we showed how to construct C_{AF} and C_{BF}. If these two relative centers sit at the same point then so does C_{AB}. On the other hand, if C_{AF} and C_{BF} do not sit at the same point then C_{AB} can be constructed by the rule of four:

$$C_{AB} = (C_{AF} \vee C_{BF}) \wedge (C_{AD} \vee C_{BD})$$

where $C_{AD} = v_1$ and where C_{BD} has been constructed in the previous paragraph as well. Notice that C_{BF} lies on the line v_2v_{n+1}, while C_{AF} lies on v_1v_{n+1}. Since in a non-degenerated realization v_1, v_2, v_{n+1} are not collinear, v_1 cannot lie on $C_{AF} \vee C_{BF}$, which guarantees that the rule of four does not degenerate.

- Next, let P be a body of the third class and A be any body of the first or the second class. In order to find the location of C_{AP} we will call for the Baracs construction. To this end we take two additional bodies of the first or the second class, B and C. By the previous paragraphs we can assume the availability of C_{AB}, C_{AC}, C_{BC}, necessarily lying on one line. Furthermore, B and C can be chosen such that these three relative centers do not sit at the same point.
  Next, we delete an edge $X \in \{D, E, F\}$ from G_{n+1} such that the resulting 2-underbraced graph G^-_{n+1} still contains P, A, B and C as bodies. Likewise the construction of C_{AE} where A belongs to the second class and E to the first, the swap principle yields the construction of the relative center lines l_{AP}, l_{BP}, l_{CP}. This situation exactly matches the elements necessary for the Baracs construction (Section 6): construct (the positions of) C_{AP}, C_{BP} and C_{CP}, fulfilling the conditions:

$$C_{AB} \in C_{AP} \vee C_{PB},$$
$$C_{AC} \in C_{AP} \vee C_{CP},$$
$$C_{BC} \in C_{BP} \vee C_{CP}.$$

- Finally, let P and Q be two bodies in G_{n+1} of the third class. Let A and B be bodies of the first or second class. Then we apply the rule of four:

$$C_{PQ} = (C_{AP} \vee C_{AQ}) \wedge (C_{BP} \vee C_{BQ}).$$

  If for each choice of $\{A, B\}$ this rule of four degenerates then one can check that this does not occur if one replaces P or Q by an appropriate R. With C_{PR} and C_{QR} available, we can now construct C_{PQ} by an alternative rule of four:

$$C_{PQ} = (C_{PR} \vee C_{QR}) \wedge (C_{BP} \vee C_{BQ})$$

  which is guaranteed non-degenerated.

Case 3. $\{v_1, v_3\}$ or (exclusively) $\{v_2, v_3\}$ share the same body in G_n.

Let us assume that there exists a body A in G_n which contains v_1 and v_3, but not v_2. So, A is still a body in $G^-_n = G_n - Z$. Consequently, $A^+ = A + \{D, F\}$ is a body in G_{n+1} (simple extension). See Fig. 12. We will treat the bodies of G_{n+1} in three different classes:

first class: the bodies A^+ and E;
second class: the bodies incident with v_2 but different from E;
third class: the remaining bodies.

- Clearly, $C_{A^+E} = v_{n+1}$ (Fig. 12).
- Let B be a body in G_{n+1} of the second class. Of course, $C_{BE} = v_2$.
  By induction, we can construct the relative center line l_{AB} in the 2-underbraced $G^-_n = G_n - Z$ (Property 5). If l_{AB} is not determined, or equivalently, if A and B belong to a common 1-underbraced subframework of G^-_n, then $C_{AB} = C_{A^+B}$ is available by induction. So, we assume that l_{AB} is well defined. Notice that the relative center line l_{A^+B} in the 2-underbraced $G_{n+1} - E$ is identical to l_{AB}.

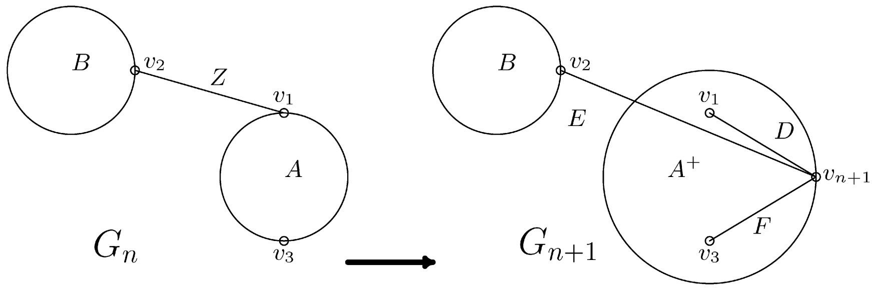

Fig. 12. Case 3: The new vertex v_{n+1} is connected to two vertices which share a common body A in G_n (v_1 and v_3), and to a third vertex which does not belong to A.

So, we find

$$C_{A^+B} = l_{A^+B} \wedge v_2 v_{n+1}.$$

This operation is well defined. Indeed, l_{AB} contains C_{AB} in G_n and C_{AB} sits at v_1v_2. But C_{AB} does not sit at v_2 (because $v_2 \neq v_1$). Furthermore, in a non-degenerated realization $v_{n+1} \notin v_1v_2$, whence C_{AB} does not sit at v_2v_{n+1}. This implies that $l_{AB} \neq v_2v_{n+1}$.

– Next, let P be a body of the third class in G_{n+1} and let B be a body of the second class. Notice that P has to be a body in G_n^- and G_n as well. By induction, we assume to have constructed l_{AP} and l_{BP} in the 2-underbraced graph G_n^- (Property 5). Furthermore, we may assume the construction of l_{EP} in the 2-underbraced $G_n^- + E + D$ due to the swap principle. As in Case 2 we find ourselves in the following situation: we are given the three relative center lines l_{AP}, l_{BP} and l_{EP}, and we know that

$$C_{A^+P} \in l_{AP}, \qquad C_{BP} \in l_{BP}, \qquad C_{EP} \in l_{EP}.$$

We have also constructed the centers C_{A^+B}, $C_{A^+E} = v_{n+1}$ and $C_{BE} = v_2$, which are not coincident in generic position. An application of the Baracs construction gives rise to C_{A^+P}, C_{BP}, C_{EP}.

– Finally, let P and Q be two bodies of the third class in G_{n+1}. Their relative center C_{PQ} can be constructed as in Case 2.

Now we combine the geometric constructions of this section with those of Section 5 (Theorem 7):

Theorem 8. *For a non-degenerated* 1*-underbraced framework F in the plane, it is always possible to obtain the relative centers of motion* (*for pairs of bodies in F*) *by means of point-line* (synthetic) *constructions. Furthermore, the only basic rules that we need are*: *the rule of four, the swap principle and the Baracs construction.*

Remarks.

– We conjecture that “non-degenerated” in Theorem 8 can be replaced by “generic.”
– As pointed out in Section 2 the polar form of the constructions in this paper can serve to construct the “lines of intersection” in the dual problem of lifting line incidences.

8. Special positions of 2-isostatic graphs

In this section we describe an important application of the synthetic constructions for relative centers of motion in 1-underbraced frameworks.

Each isostatic graph G has "bad" realizations, which means that the bars of the constructed (planar) framework F are not independent, and F becomes infinitesimally flexible. These are called *special positions* of the isostatic graph G. In general, a special position of a structure graph G is any realization with "decreased rank," causing bar dependencies that are not predicted by Laman's count (nongeneric dependencies). Special positions with coinciding joints are regarded as uninteresting or improper, and are called *degenerate positions*. In [17] the authors prove that the non-degenerate special positions of an isostatic graph G can be described as the zero set of one polynomial $C(G)$, called the (*pure*) *condition* of G. Specifically, if an isostatic graph G has v vertices, and $e = 2v - 3$ edges, then $C(G)$ is a homogeneous polynomial in $3v$ variables, representing the homogeneous coordinates (a_1, a_2, a_3) of each vertex a. Furthermore, because infinitesimal rigidity is a projective invariant, $C(G)$ can be expressed in the *Bracket Ring*, which means that it can be written as a polynomial in "brackets" $[abc]$ of vertices $a, b, c, \ldots,$ with

$$[abc] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Each term of $C(G)$ is a product of $v - 2$ such brackets, and the degree of each vertex a equals $\mathrm{val}(a) - 1$, where $\mathrm{val}(a)$ denotes the *valence* of a in G (the number of incident edges). There are known procedures to obtain the pure condition $C(G)$ of a given isostatic graph G, e.g. see the algorithm in [12].

Though $C(G) = 0$ gives the complete algebraic description for the locus of special positions of G, it fails to provide geometric insight. It is the purpose of this section to give a general procedure to derive a join/meet-expression for a given isostatic graph G, of which the zeros yield special positions of G.

Let G be a given isostatic graph with edge ab. If we delete ab from G, we obtain a graph $G' = G - ab$, that can be realized as a 1-underbraced framework $F' = (G', p)$ in the plane. This means that for each pair of bars (or bodies) A and B in F' the relative center of motion C_{AB} sits at a determined point in the projective plane, not depending on the applied motion (unless A and B share the same body in F', implying $C_{AB} = 0$). Furthermore, in the previous sections we gave a synthetic construction for these relative centers. Notice that the construction of C_{AB} can be given as a join/meet expression in the joints of F', using maybe one additional generic point for drawing an auxiliary line each time that we call the Baracs construction. Also notice that this construction can be described in terms of the vertices of G, independent from the realization p, yielding an expression Φ_{AB} (of rank 1) in the Grassmann Algebra, with the vertices of G as variables, and maybe one additional variable to represent a generic point. This expression Φ_{AB} is only identically zero if A and B share a common body in G'. Of course, there might be still certain realizations p in which the construction of C_{AB} degenerates, that is, $\Phi_{AB}(p) = 0$. This means that during the construction we connect two coinciding points, or intersect two coinciding lines. Consequently, C_{AB} is not unambiguously determined anymore. By Theorem 2, this can never occur in a generic position p (for which $F' = (G', p)$ is 1-underbraced).

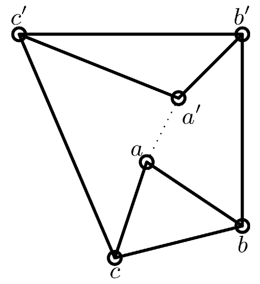

Fig. 13. If we omit bar aa', the framework becomes 1-underbraced in almost every realization. Special positions occur if a triangle degenerates, or if b, b', c, c' become collinear. In a non-special position, the relative center for the two triangles can be obtained by the "rule of four."

Examples.

- Let G consist of two triangles, abc and $a'b'c'$, that are connected by three edges aa', bb', cc' (Fig. 13). This is a well-known structure graph for an isostatic framework in the plane ($e = 2v - 3 = 9$). So, if we omit one edge, aa' say, we obtain a 1-underbraced graph G'. Suppose we want to construct the relative center C_{AB} for the bodies $A = abc$ and $B = a'b'c'$. In almost every realization p of G' this construction can be represented by the formula:

$$\Phi_{AB} = (b \vee b') \wedge (c \vee c') = bb' \wedge cc' = [bb'c]c' - [bb'c']c.$$

  In special positions where the triangles A and/or B degenerate, and hence cease to be bodies, we replace A and/or B by an edge of the triangle, and C_{AB} makes sense again, and is still represented by the same expression Φ_{AB}. On the other hand, if the realization p specializes the framework such that b, b', c, c' are collinear, the construction for C_{AB} collapses, since the intersection of two coinciding lines is not determined. This is reflected in the fact that $\Phi_{AB}(p)$ becomes zero in the Grassmann Algebra. However, in this position, $F' = (G', p)$ is at least 2-underbraced, so the failure for the construction of C_{AB} makes sense.
- If we take $G = K_{3,3}$, then we get another famous 2-isostatic graph with 6 vertices (Fig. 14). Once again we delete edge aa', and obtain a 1-underbraced graph G'. Let us consider

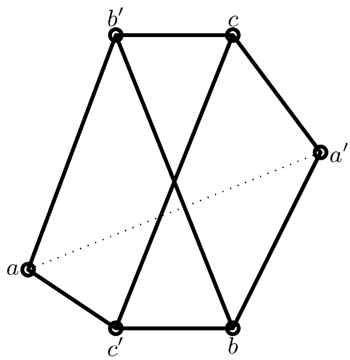

Fig. 14. If we delete edge aa' then we can realize the graph as a 1-underbraced framework. In certain special positions, for example when a, b', c, c' are collinear, the framework becomes at least 2-underbraced.

the construction for the relative center $C_{ab',a'b}$. Using the join/meet-formulas for $C_{ab',cc'}$ and $C_{a'b,cc'}$,

$$\Phi_{ab',cc'} = (a \vee c') \wedge (b' \vee c) = ac' \wedge b'c,$$
$$\Phi_{a'b,cc'} = (a' \vee c) \wedge (b \vee c') = a'c \wedge bc'$$

we find that

$$\Phi_{ab',a'b} = b'b \wedge \big((ac' \wedge b'c) \vee (a'c \wedge bc')\big).$$

Notice that $\Phi_{ab',cc'} = 0$ (respectively, $\Phi_{a'b,cc'} = 0$) if the line ab' (respectively, the line $a'b$) is coincident with cc'. Other special positions in which $\Phi_{ab',a'b}$ vanishes, occur if $ac' \wedge b'c$ coincides with $a'c \wedge bc'$, or if both points are collinear with b and b'.

Theorem 9. *Let G be an isostatic graph with edge ab, and let G' be the* 1*-underbraced graph obtained from G by deleting ab. Let A be an edge* (*or body*) *of G' that contains vertex a, and let B be an edge* (*or body*) *of G' that contains vertex b. Furthermore, let Φ_{AB} be the* (*generic*) *join/meet-formula for the relative center of A and B in G'. Then, the zeros of*

$$\Phi_{AB} \vee a \vee b$$

correspond to special positions p of G.

Proof. Let p be a zero of $\Phi_{AB} \vee a \vee b$. Notice that Φ_{AB} is not identically zero, since A and B do not belong to a common body of G' (otherwise ab would be a generically implied edge in G, which is excluded by the fact that G is isostatic).

If $F' = (G', p)$ is at least 2-underbraced then the motion space of this framework has at least 5 dimensions. Since $F = (G, p)$ has one additional bar w.r.t. F', its motion space has at least 4 dimensions. In this case, it is already implied that p is a special position of G. So, let us assume from now on that $F' = (G', p)$ is 1-underbraced.

This means that $\Phi_{AB}(p) \neq 0$, and that for each non-trivial motion γ of F' the relative center C_{AB} is a multiple of $\Phi_{AB}(p)$. This implies that always $C_{AB} \vee a \vee b = 0$, whence that bar ab is implicit in F'. Consequently, the bars of $F = (G, p)$ are not independent. And so p is a special position of G. □

Remark. Although the formula $\Phi_{AB} \vee a \vee b$ encodes special positions of the isostatic graph G, this need not be all the special positions. So, $\Phi \vee a \vee b$ is not necessarily equal to the pure condition $C(G)$. Below, examples are presented where our formula is rather a factor of $C(G)$. Furthermore, if Φ_{AB} describes a non-intrinsic construction, then it contains a generic point z, which does not correspond to a joint of the framework. In this case, $\Phi_{AB} \vee a \vee b$ cannot be equal to $C(G)$.

Examples.

– Let G be the isostatic graph of Fig. 13, two triangles $A = abc$ and $B = a'b'c'$, connected by three bars aa', bb', cc'. We already know Φ_{AB} for $G \setminus aa'$. From this we obtain:

$$\Phi_{AB} \vee a \vee a' = bb' \vee cc' \vee aa'$$

which becomes zero in (non-degenerate) realizations of G where the triangles A and B are in point perspective (lines aa', bb' and cc' are concurrent). On the other hand, if we take $G' = G \setminus ab$, $A = ac$ and $B = bc$, then we find special positions described by

$$\Phi_{AB} \vee a \vee b = c \vee a \vee b = 0$$

which exactly occurs when joints a, b, c are collinear. In a similar way, we see special positions associated with $c' \vee a' \vee b' = 0$. If we expand into brackets, then we get:

$$\begin{aligned} aa' \vee bb' \vee cc' &= [aa'b][b'cc'] - [aa'b'][bcc'], \\ a \vee b \vee c &= [abc], \\ a' \vee b' \vee c' &= [a'b'c']. \end{aligned}$$

In [17] the authors derive for this graph G the pure condition

$$C(G) = \big([aa'b][b'cc'] - [aa'b'][bcc']\big)[abc][a'b'c']$$

which exactly matches our findings.

– Let $G = K_{3,3}$ as in Fig. 14. Earlier, we derived the formula Φ_{AB} in $G' = G \setminus aa'$, with $A = ab'$ and $B = a'b$. So,

$$\Phi_{AB} \vee a \vee a' = \big[b'b \wedge \big((ac' \wedge b'c) \vee (a'c \wedge bc')\big)\big] \vee aa'$$

which becomes zero if $aa' \wedge bb'$, $ac' \wedge b'c$ and $a'c \wedge bc'$ are collinear. By Pascal's Theorem this exactly occurs if the six joins lie on one conic. If we check the degrees of the joint variables in this expression, we conclude that it is the Cayley factorization of $C(G)$. Indeed, expanding it into brackets gives us

$$\begin{aligned} &[ac'b'][a'cb'][b'cc'][baa'] - [ac'b'][a'cb'][bcc'][b'aa'] \\ &\quad + [ac'c][a'cb][bb'c'][b'aa'] - [ac'b'][a'cc'][b'cb][baa'] \end{aligned}$$

which is equivalent to the formula as stated in [17].

– Although the method of Theorem 9 provides a geometric interpretation for the pure condition $C(G)$ of the graphs in the previous two examples, no new result was obtained. Indeed, the Cayley factorization of $C(G)$ for these graphs has been known for a long time (see e.g. [17]). Let us now give an example of which no join/meet-formula for $C(G)$ has appeared in the literature yet. To this end we start with the graph K_{33} of Fig. 14, and extend it by means of the following Henneberg move of type 2: delete edge ab', rename a into a_2, and add a 3-valent vertex a_1 linked with a_2, b' and a' (Fig. 15).

  We consider the 1-underbraced graph $G' = G \setminus cc'$. If Δ denotes the triangle a_1a_2a', and after applying the rule of four twice, we obtain

$$\begin{aligned} C_{bc',\Delta} &= a'b \wedge a_2c', \\ C_{b'c,\Delta} &= a'c \wedge a_1b' \end{aligned}$$

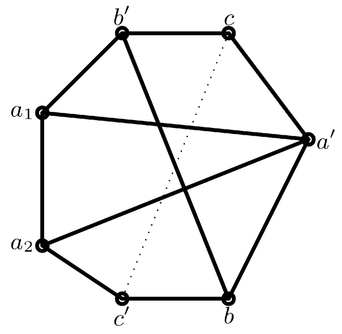

Fig. 15. If the relative center of bars bc' and $b'c$, considered in the 1-underbraced graph $G \setminus cc'$, is realized on the line cc', then the complete framework $F = (G, p)$ becomes infinitesimally flexible.

and from this,

$$\begin{aligned} C_{bc',b'c} &= bb' \wedge (C_{bc',\Delta} \vee C_{b'c,\Delta}) \\ &= bb' \wedge \big((a'b \wedge a_2c') \vee (a'c \wedge a_1b')\big). \end{aligned}$$

Consequently, in G' the formula $\Phi_{bc',b'c}$ is given by the RHS of the last equation. We conclude that a framework F with the given graph G as structure graph is in special position if

$$cc' \vee \big[bb' \wedge \big((a'b \wedge a_2c') \vee (a'c \wedge a_1b')\big)\big] = 0$$

which exactly occurs if $bb' \wedge cc'$, $a_2c' \wedge a'b$ and $a_1b \wedge a'c$ are collinear. Notice that this comes down to the pure condition of K_{33} after substituting $a_1 = a_2 = a$. Expanding into brackets leads us to

$$\begin{aligned} &[a'ca_1][a'ba_2][c'b'b][b'cc'] - [a'bc'][a_2b'b][a'ca_1][b'cc'] \\ &\quad - [a'ba_2][a'cb'][c'a_1b][b'cc'] + [a'ba_2][a'cb'][c'a_1b'][bcc'] \\ &\quad + [a'bc'][a'cb'][a_2a_1b][b'cc'] - [a'bc'][a'cb'][a_2a_1b'][bcc']. \end{aligned}$$

If we multiply this polynomial by the factor $[a_1a_2a]$, referring to the special positions of Δ, we get the right degrees. So, this product must be equal to $C(G)$.

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